DISTRIBUTION OF PRESENT VALUE
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Abstract

This paper presents the distribution of t he present value of a project or asset under different processes for future payoffs. For all hypothesized processes, the coefficient of variation (rela tive uncertainty) is proporti onal to the coefficient of variation of the nex t payoff, the factor of proportionality be ing function of the discount rate. The variability of the present value may increase or decrease with respect to the discount rate, depending on the type of process. It is higher when payoffs follow a random walk than for the hypo thesized processes with independent payoffs, and becomes extremely large for low discount rates.

Key words: Coefficient of variation, discounting, present value.
JEL Classification: D80, G12.

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1. Introduction

Present value relationship, one of the ma in economic relationships, constitutes the core of many economic theories and models. It is ubiquitous , and is us ed throughout the valuation of economic projects and a variety of assets, such as companies, stocks, real state and even human lives. The origin of this relationship is very remote, trac ing back to the beginning of the 13 [th] century (see Goetzmann, 2004).

This relationship equals the value of the project or asset at a certain moment in time, t, to the total of discounted future net payoffs:

$$P_t = \sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i} \qquad (1.1)$$

where P_t is the net present value in t, $D_{t+1}, D_{t+2}, \dots$ are net payoffs corresponding to moments $t+1, t+2, \dots$ and $r > 0$ is the discount rate.

Though future payoffs are often unsure, their uncertainty is hardly ever taken into account and the contributions that ex plicitly deal with the random nature of the present value relationship are very limited Very often only t heir expected values are used and, consequently, only th e expectation of the present value of these payoffs is obtained. One clear example may be found in investment analysis or capital budgeting. As stated in Lee and Tai (2013), three alternativ e methods are used: statistical, decis ion-tree and s imulation methods. Thes e methods do not allow obtaining t he statistical distribution of the present value; even the model proposed by Hillier (1 963) presents an ad-hoc treatment of uncertainty of future payoffs (see also Chen and Moore (1982) or Rajaratnam et al. (2014) for a recent appl ication of certainty equiv alent method). However, given their uncertain nature, they ma y be represented by random variables wit h certain characteristics. Then, if the di scount rate is cons idered deterministic, (1.1) expresses the present value as a combination of r andom variables with decreasing weights. This paper aims to study the distribution of P_t under different stochastic models for future payoffs . Two features of the distribution of P_t will be specia lly interesting: the expectation and the variance (or,

equivalently, the standard devia tion or the coefficient of variation). They will provide measures of the expected value and the risk, respectively. In addition, it will also be interesting to analyz e the relationship bet ween these two features and the discount rate, r. To achieve thes e objectives, Section 1 proposes different statistical distributions for future payoffs, Section 2 obtains the distributions of P_t under these distributions, and Section 3 comments on their properties.

2. Different models for future payoffs

Future payoffs may follow very different random distributions. While so me distributions are rather unrealistic for mo st projects or assets, others could be reasonably hypothesized in certain situations. In what follows, the presumptions made will be based on the plau sibility and mathematical tractability of the distributions. A first convenient assumption will be their normality, as the lin ear combination of normally distributed payoffs will yield a distribution also normal for P_t.[1] These normal distributions are det ermined by their expec tations and variances. Initially, expectations will be considered constant over time for the different payoffs, while variances will be constant or increasing over time, thus reflecting the fact that more distant pay offs are usually more uncertain. Thes e assumptions permit us to consider the following class of mod els for future payoffs:

$$D_{t+i} \sim N(\overline{D},\ i^{\alpha}\sigma^2),\ i = 1, 2, \ldots \tag{2.1}$$

with D_{t+i} and D_{t+j} being independent random variables if $i \neq j$, $\overline{D} > 0$ and $\alpha \geq 0$. The parameter α regulates the variability of future payoffs. Taking $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$, the following respective processes are hypothesized:

Process 1 (P1): $D_{t+i} \sim N(\overline{D},\ \sigma^2),\ i = 1, 2, \ldots$
Process 2 (P2): $D_{t+i} \sim N(\overline{D},\ i\sigma^2),\ i = 1, 2, \ldots$
Process 3 (P3): $D_{t+i} \sim N(\overline{D},\ i^2\sigma^2),\ i = 1, 2, \ldots$

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[1] In any case, central limit theorems may be invoked when other distributions are considered for future payoffs.

P1 follows from (2.1) with $\alpha = 0$. The uncertainty of future payoffs (measured by their standard deviations or variances) will be constant. Uncertainty does not increase with the time interval to the moment of payoff. When $\alpha = 1$ in (2.1), one obtains P2. In this model, uncertainty increases with the time interval to the payoff; the variances increase propor   tionally, and the standard deviations increase less than proportionally . That is, the varianc e of payoff in $t + n$ is n times the variance of payoff in $t + 1$, and the standard devia tion of payoff in $t + n$ is $\sqrt{n}$ times the standard deviation of payoff in $t + 1$ for any $n > 1$. P3 follows from (2.1) with $\alpha = 2$. In this case, the standard dev iation is proportional to the time interval (a payoff in $t + n$ has a standard deviat ion equal to n times the standard deviation of payoff in $t + 1$). In all three cases the unc ertainty of the next payoff is the same, but the uncertainty of payoffs that will take place in $t + n$, for any $n > 1$, will increase with α.

The independence assumption between the different payoffs may be relaxed by supposing that they follow a random walk,

Process 4 (P4): $D_{t+i} = D_{t+i-1} + \varepsilon_{t+i}, \; i = 1, 2, ...$

with $D_t = \bar{D} > 0$, $\varepsilon_{t+i} \sim N(0, \sigma^2)$ for all i, and $Cov(\varepsilon_{t+i}, \varepsilon_{t+j}) = 0$ for any $i, j > 0, \; i \neq j$. In this model the e xpected value of every future payoff will be constant, $\bar{D}$. As $Var(D_{t+i}) = i\sigma^2$, their variances   will increase with the time horizon to the payoff date, and will be equal to those under P2 (pr oportional to the time i nterval). This proces s P4 may also be extend   ed by allowing the expectations of future payoffs to grow at rate g:

Process 5 (P5): $D_{t+i} = (1 + g)D_{t+i-1} + \varepsilon_{t+i}, \; i = 1, 2, ...$

with $D_t = \bar{D}/(1 + g) > 0$, $0 < g < r$, $\varepsilon_{t+i} \sim N(0, \sigma^2)$ for all i,   and $Cov(\varepsilon_{t+i}, \varepsilon_{t+j}) = 0$ for any $i, j > 0, \; i \neq j$. In this model g is the non-random rate of growth of the expec tations of future payoffs. If $g = 0$, P4 would be obtained. As $g > 0$, both the exp ected value of payoffs and their uncertainty will increase with the time horizon. The   expectation and the standard deviation of the first payoff, D_{t+1}, are the sam e for P5 a s for the other processes P1-P4. Table 1

shows the standard deviations of differ ent payoffs under P1-P4, and under P5 for different values of g.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=10$	$i=20$	$i=50$
P1	1	1	1	1	1	1	1	1
P2	1	1.41	1.73	2	2.24	3.16	4.47	7.07
P3	1	2	3	4	5	10	20	50
P4	1	1.41	1.73	2	2.24	3.16	4.47	7.07
P5 ($g=3\%$)	1	1.44	1.79	2.09	2.38	3.64	6.09	17.30
P5 ($g=10\%$)	1	1.49	1.92	2.33	2.75	5.22	14.52	256.16

Table 1. Standard deviations of se veral future payoffs D_{t+i}, expressed as times th e standard deviation of payoff D_{t+1} (σ), under processes P1-P5.

3. Distributions of present value under the different models for future payoffs

Under P1-P5, P_t is normally distributed, given t hat future payoffs are normally distributed. These normal distributions ar e fully determined by their respectiv e expectations and variances, which are stated in the following propositions.

3.1. Under P1,

$$P_t \sim N\left(\frac{\bar{D}}{r}, \frac{\sigma^2}{(1+r)^2-1}\right). \tag{3.1}$$

(See Appendix A1).

3.2. Under P2,

$$P_t \sim N\left(\frac{\bar{D}}{r}, \frac{(1+r)^2}{((1+r)^2-1)^2}\sigma^2\right). \tag{3.2}$$

(See Appendix A2).

3.3. Under P3,

$$P_t \sim N\left(\frac{\bar{D}}{r}, \frac{(1+r)^4+(1+r)^2}{((1+r)^2-1)^3}\sigma^2\right). \tag{3.3}$$

(See Appendix A3).

3.4. Under P4,

$$P_t \sim N\left(\frac{\overline{D}}{r}, \frac{(1+r)^2\sigma^2}{((1+r)^2-1)^2} + \frac{2\sigma^2}{r((1+r)^2-1)} + \frac{2\sigma^2}{r((1+r)^2-1)^2}\right). \qquad (3.4)$$

(See Appendix A4).

3.5. Under P5,

$$P_t \sim N\left(\frac{\overline{D}}{r-g}, \frac{(1+r)^2\sigma^2}{((1+r)^2-1)((1+r)^2-(1+g)^2)} + \right.$$

$$+ \frac{2(1+g)^3\sigma^2}{(r-g)((1+g)^2-1)((1+r)^2-(1+g)^2)} -$$

$$\left. - \frac{2(1+g)\sigma^2}{(r-g)((1+g)^2-1)((1+r)^2-1)}\right). \qquad (3.5)$$

(See Appendix A5).

4. Properties of the distributions of present value

Under P1-P4, it is clear that $E(P_t)$ is decreasing and is strictly convex with respect to r. $E(P_t)$ tends to $+\infty$ (0) when r tends to 0 $(+\infty)$. See Fig. 1. Changes in r provoke higher ab solute changes in $E(P_t)$ when r is low than when r is high. This does not apply to rela tive changes; in fact, a relative change in r equal to λ implies a relative change in $E(P_t)$ equal to $-\lambda/(1+\lambda)$ for any r. Under P5 the expectation of P_t is:

$$E(P_t) = \frac{\overline{D}}{r-g}$$

an expression known as 'Gordon growth model'. When g is close to r, $E(P_t)$ will be high and, furthermore, chang es in any of these v ariables will hav e strong effects on the expectations of P_t.
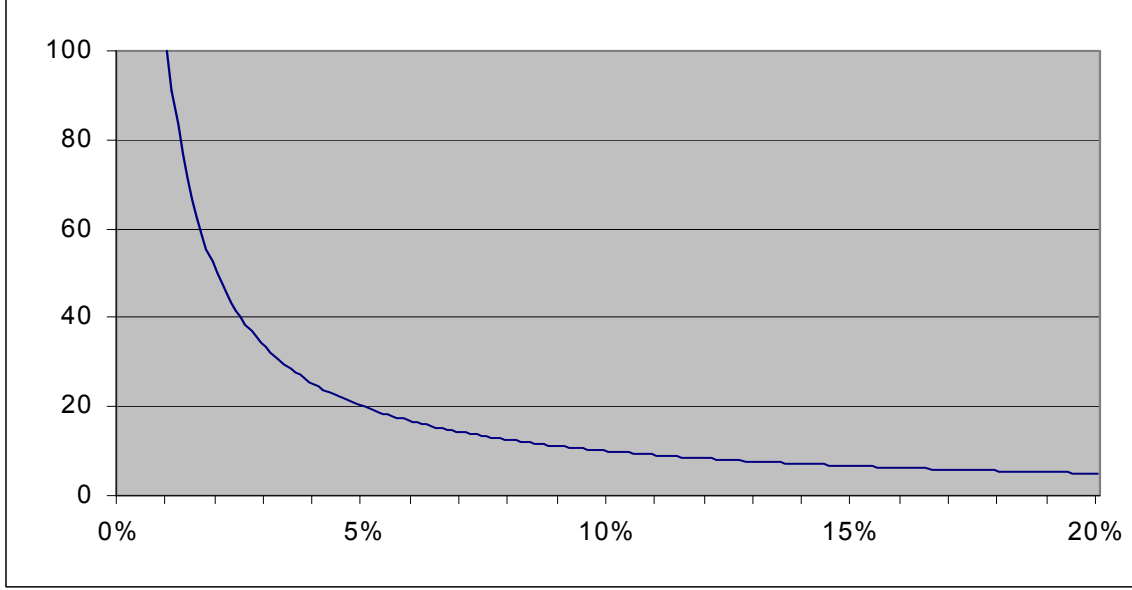
Fig. 1. Expectation of the present value $E(P_t)$, expressed as times the expectation of the next payoff $(\overline{D})$, against the discount rate r under processes P1-P4.

A relative measure of risk or uncertainty of P_t will be its coefficient of variation, its standard deviation divided by its expect ation. Under P1-P4 t he expectations of P_t are directly proportional to $\overline{D}$, and their standard deviations are directly proporti onal to σ. Given these proportionality relations, the coefficients of variation of P_t will be directly proportional to the coefficient of variation of the next payoff, $\sigma/\overline{D}$, and then may be written as:

$$CV_i = \frac{\sigma}{\overline{D}} f_i(r), \;\; i = 1, 2, 3 \text{ and } 4 \qquad (4.2)$$

where CV_1, CV_2, CV_3 and CV_4 denote the coefficients of variation of P_t under processes P1, P 2, P3 and P4, respectively, and $f_i(r)$, with $i = 1, 2, 3$ and 4, denote different functions with r as the only argument.

Table 2 and Fig. 2 show the coeffi cients of variation as functions of r expressed as times the ratio $\sigma/\overline{D}$. The selected range for r intends to cover a range of plausible values (from 0% to 20%), though the disparit y in values for the discount rate in previous res earch is enormous (see Fre derick et al., 2002). Under the different processes, the coeffi cients of variation present the following properties:

7

i) Under P1 the coefficient of vari ation of P_t increases with respect to r. However, it is relatively low, ranging appr oximately from 0 to 0.3 times the ratio $\sigma/\overline{D}$ for $0 < r < 20\%$. Thus, taking, for example, $\sigma/\overline{D} = 0.1$, a ratio which could be a reas onable value in certain c ircumstances, the coefficient of variation moves approximately from 0 to 3% when r changes from 0 to 20%.

ii) Under P2 the coefficient of variation of P_t is almost constant, ranging approximately from 0.50 to 0.55 times the ratio $\sigma/\overline{D}$ for $0 < r < 20\%$. This implies that, for example, for $\sigma/\overline{D} = 0.1$ the coeffi cient of variation moves approximately from 5.0% to 5.5% when r changes from 0 to 20%.

iii) Under P3 the coefficient of variation of P_t is much higher than under P1 or P2, and decreases with respect to r. The diminution is important. For example, for $r = 1\%$ the coefficient of variation is approximately 5 times the ratio $\sigma/\overline{D}$, while for $r = 5\%$ it is about 2.3 times. As reflected in Fig. 2, when r is high, changes in r have a small effect on the c oefficient of variation of P_t, but when r is low, they have a large effect.

iv) Under P4 the coefficient of variation of P_t is higher than under P1-P3. Its behavior is similar to that found under P3. It also decreases with respect to r, and the decrement is very pr ominent for low values of r. When r is hig h, changes in r have a much more limited effect on the coefficient of variation of P_t.

	$r = 1\%$	$r = 3\%$	$r = 5\%$	$r = 10\%$	$r = 15\%$	$r = 20\%$
P1	0.07	0.12	0.16	0.22	0.26	0.30
P2	0.50	0.51	0.51	0.52	0.53	0.55
P3	5.04	2.95	2.32	1.70	1.44	1.28
P4	7.12	4.17	3.28	2.40	2.03	1.81

Table 2. Coefficients of variation of the present value, expressed as t imes the coefficient of variation of the next pa yoff, $\sigma/\overline{D}$, for different values of the discount rate r under P1-P4.
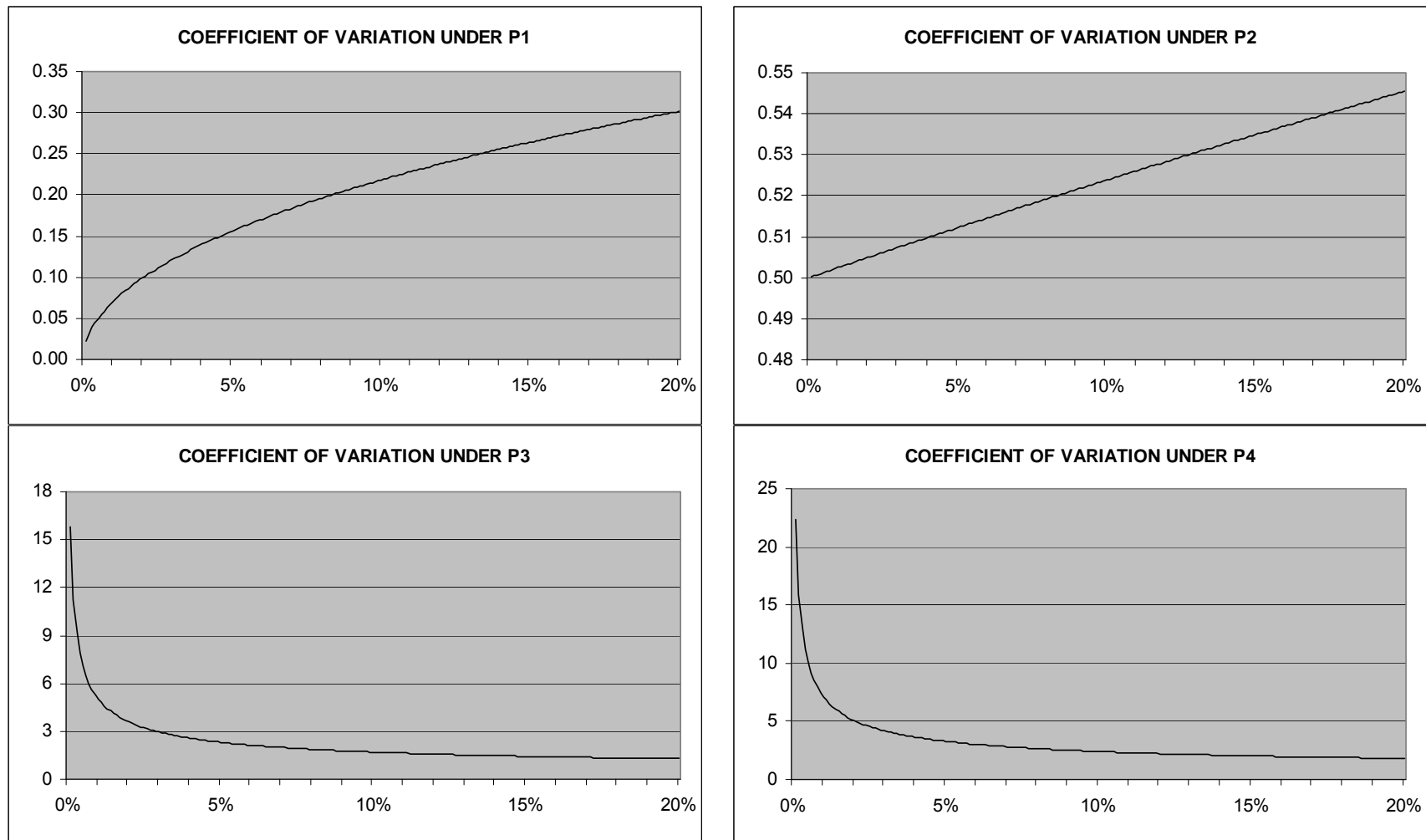
Fig. 2. Coefficients of variation of the present value, expressed as times the coefficient of variation of the next payoff $(\sigma/\bar{D})$, against the discount rate r under processes P1-P4.

The standard deviation of future independent payoffs, D_{t+i}, with $i > 1$, is greater under P3 than under P2, and greater under P2 than under P1; therefore, for a given r, $CV_1 < CV_2 < CV_3$. It should be noted that the relative difference depends on r. Thus, for example, for $r = 20\%$, the coefficient of variation under P3 is a little more t han double the coefficient under P2, but for $r = 1\%$, it is more than tenfold. It is also interesting to compare CV_2 and CV_4. The expectations and standard deviations of future payoffs are exactly equal under these models. However, the variance of P_t under P2 is:

$$\frac{(1+r)^2\sigma^2}{((1+r)^2 - 1)^2} \tag{4.3}$$

but under P4 it is

$$\frac{(1+r)^2\sigma^2}{((1+r)^2 - 1)^2} + \frac{2\sigma^2}{r((1+r)^2 - 1)} + \frac{2\sigma^2}{r((1+r)^2 - 1)^2} \tag{4.4}$$

One can observe that (4.3) is the first summand of (4.4). It is the part of the variance of P_t that is due to the variances of future payoffs. The difference between (4.4) and (4.3),

$$\frac{2\sigma^2}{r((1+r)^2 - 1)} + \frac{2\sigma^2}{r((1+r)^2 - 1)^2}$$

is the component of the variance of P_t under P4 due to the co-variability o f future payoffs. As this second com ponent is po sitive, the variance and, therefore, the coefficient of variation of P_t will always be higher under P4 than under P2. In fact, for typical values of r, the second component due to the co-variability is much larger than the firs t component d ue to the variability, and therefore CV_4 is much hi gher than CV_2, as shown in Fig. 2 for values of r comprised between 0% and 20%,

Let us now examine the coefficient of variation of the present value under P5, CV_5. As both the expectation and the standard deviation of the present value depend on both r and g, one could think that CV_5 also depends on both r and g. However, the following propositio n states that it only depends on r, and that it is equal to that under P4.

Proposition 4.1 Under P5, the coefficient of variation of the present value is:

$$\frac{r}{\overline{D}}\sqrt{\frac{(1+r)^2\sigma^2}{((1+r)^2-1)^2}+\frac{2\sigma^2}{r((1+r)^2-1)}+\frac{2\sigma^2}{r((1+r)^2-1)^2}}$$

Proof. It is easy (but rather cumbersome)   to prove that the quotient of the variance and the square of   the expectation of the pr esent value under P5 is equal to that under P4. Thus, equality of the coefficients of variation follows.

Therefore, under P5, g determines the expectatio ns and the standard deviations of future payoffs. It also determines the expectation and the standard deviation of the present value. However, it does not determine the coefficient of variation of the present value, which is equal to that under P4, $CV_5 = CV_4$.

5 Conclusions

This paper provides the di stribution of the present va lue under different feasible processes for future payoffs, with specia l attention paid to its relative variab ility measured   by the coefficient of variat   ion. The results obt   ained with the hypothesized processes show that the re   lationship of the variability of the present value with the discount   rate depends on the process of the future payoffs taken into account. While pr   ocesses with independ ent payoffs of constant or slowly increasing uncert ainty may present an increasing (though limited) variability of the pr esent value with respect to the disc   ount rate, the opposite pattern may occur when the uncer tainty of the payoffs increases more quickly. Processes with dependent payoffs   present a higher variability of the present value due t o   the co-variation of   future payoffs, and this clearly decreases with respect to the discount ra te. When con sidering extensions of a random walk for future payoffs that also   allow expec tations to increase ov er time at a given rate, it is found that t   he coefficient of vari ation of the present value does not depend on the rate of growth.

Appendix A1

$$E(P_t) = E\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-i}E(D_{t+i}) = \bar{D}\sum_{i=1}^{\infty}(1+r)^{-i} = \frac{\bar{D}}{r}.$$

$$Var(P_t) = Var\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-2i}Var(D_{t+i}) = \sigma^2\sum_{i=1}^{\infty}(1+r)^{-2i} = \frac{\sigma^2}{(1+r)^2-1}.$$

Appendix A2

$$E(P_t) = E\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-i}E(D_{t+i}) = \bar{D}\sum_{i=1}^{\infty}(1+r)^{-i} = \frac{\bar{D}}{r}.$$

$$Var(P_t) = Var\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-2i}Var(D_{t+i}) = \sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i}.$$

$\sum_{i=1}^{\infty}i(1+r)^{-2i}$ is an arithmetic-geometric series and then:

$$\sum_{i=1}^{\infty}i(1+r)^{-2i} = \frac{(1+r)^{-2}}{1-(1+r)^{-2}} + \frac{(1+r)^{-4}}{\left(1-(1+r)^{-2}\right)^2} = \frac{(1+r)^2}{\left((1+r)^2-1\right)^2}.$$

Therefore,

$$Var(P_t) = \frac{(1+r)^2}{\left((1+r)^2-1\right)^2}\sigma^2.$$

Appendix A3

$$E(P_t) = E\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-i}E(D_{t+i}) = \bar{D}\sum_{i=1}^{\infty}(1+r)^{-i} = \frac{\bar{D}}{r}.$$

$$Var(P_t) = Var\left(\sum_{i=1}^{\infty}(1+r)^{-i}D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-2i}Var(D_{t+i}) = \sigma^2\sum_{i=1}^{\infty}i^2(1+r)^{-2i}.$$

The series $\sum_{i=1}^{\infty}i^2(1+r)^{-2i}$ is convergent (D'Alembert criterion). Then, if

$$S_i = (1+r)^{-2} + 4(1+r)^{-4} + 9(1+r)^{-6} + \cdots + i^2(1+r)^{-2i},$$

$$\frac{S_i}{(1+r)^2} = (1+r)^{-4} + 4(1+r)^{-6} + 9(1+r)^{-8} + \cdots + i^2(1+r)^{-2i-2},$$

$$\frac{(1+r)^2-1}{(1+r)^2}S_i = (1+r)^{-2} + 3(1+r)^{-4} + 5(1+r)^{-6} + \cdots + (2i-1)(1+r)^{-2i} - i^2(1+r)^{-2i-2},$$

$$\frac{\left((1+r)^2-1\right)}{(1+r)^4}S_i = (1+r)^{-4} + 3(1+r)^{-6} + 5(1+r)^{-6} + \cdots + (2i-1)(1+r)^{-2i-2} - i^2(1+r)^{-2i-4},$$

$$\frac{(1+r)^2-1}{(1+r)^2}S_i - \frac{(1+r)^2-1}{(1+r)^4}S_i = \left(\frac{(1+r)^2-1}{(1+r)^2}\right)^2 S_i = (1+r)^{-2} + 2(1+r)^{-4} + 2(1+r)^{-6} + \cdots +$$

$$+\cdots+2(1+r)^{-2i+2}+2(1+r)^{-2i}+(i^2-2i+1)(1+r)^{-2i-2}+i^2(1+r)^{-2i-4}=$$

$$=(1+r)^{-2}+\frac{2(1+r)^{-2i-2}-2(1+r)^{-4}}{(1+r)^{-2}-1}+(i-1)^2(1+r)^{-2i-2}+i^2(1+r)^{-2i-4}.$$

As $\sum\limits_{i=1}^{\infty} i^2(1+r)^{-2i-4}$ and $\sum\limits_{i=1}^{\infty}(i-1)^2(1+r)^{-2i-2}$ are convergent given that

$$\lim_{i\to\infty}\frac{(i+1)^2(1+r)^{-2i-6}}{i^2(1+r)^{-2i-4}}=\lim_{i\to\infty}\frac{(i+1)^2}{i^2(1+r)^2}<1\text{, and}$$

$$\lim_{i\to\infty}\frac{i^2(1+r)^{-2n-4}}{(i-1)^2(1+r)^{-2n-2}}=\lim_{i\to\infty}\frac{i^2}{(i-1)^2(1+r)^2}<1\text{, then,}$$

$$\sum_{i=1}^{\infty}i^2(1+r)^{-2i}=\lim_{i\to\infty}S_i=\left(\frac{(1+r)^2}{(1+r)^2-1}\right)^2\lim_{i\to\infty}\left(\frac{1}{(1+r)^2}+\frac{2(1+r)^{-2i-2}-2(1+r)^{-4}}{(1+r)^{-2}-1}\right)=$$

$$=\left(\frac{(1+r)^2}{(1+r)^2-1}\right)^2\left(\frac{1}{(1+r)^2}+\frac{2}{(1+r)^4-(1+r)^2}\right)=\frac{(1+r)^4+(1+r)^2}{\left((1+r)^2-1\right)^3},$$

and $Var(P_t)=\dfrac{(1+r)^4+(1+r)^2}{\left((1+r)^2-1\right)^3}\sigma^2.$

Appendix A4

$$E(P_t) = E\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-i} E(D_{t+i}) = \bar{D}\sum_{i=1}^{\infty}(1+r)^{-i} = \frac{\bar{D}}{r}.$$

$$Var(P_t) = Var\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-2i} Var(D_{t+i}) + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j} Cov(D_{t+i}, D_{t+j}) =$$

$$= \sum_{i=1}^{\infty}(1+r)^{-2i}\sum_{j=1}^{i}\sigma^2 + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j}\sum_{k=1}^{i}\sigma^2 =$$

$$= \sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i} + 2\sigma^2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}i(1+r)^{-i}(1+r)^{-j}.$$

$Var(P_t)$ is expressed as the sum of two components: the first, $\sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i}$, is the part of $Var(P_t)$ due to the variability of future payoffs, while the second,

$2\sigma^2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}i(1+r)^{-i}(1+r)^{-j}$, is the part of $Var(P_t)$ due to the co-variability of future different payoffs. The first component is:

$$\sigma^2\sum_{i=1}^{\infty}i(1+r)^{-2i} = \frac{(1+r)^2}{\left((1+r)^2-1\right)^2}\sigma^2 \text{ (see Appendix A2), and the second is}$$

$$2\sigma^2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}i(1+r)^{-i}(1+r)^{-j} =$$

$$= 2\sigma^2\left((1+r)^{-1}\sum_{j=2}^{\infty}(1+r)^{-j} + 2(1+r)^{-2}\sum_{j=3}^{\infty}(1+r)^{-j} + 3(1+r)^{-3}\sum_{j=4}^{\infty}(1+r)^{-j} + \cdots\right) =$$

$$= 2\sigma^2\left(\frac{1}{r(1+r)^2} + \frac{2}{r(1+r)^4} + \frac{3}{r(1+r)^6} + \cdots\right) = \frac{2\sigma^2}{r\left((1+r)^2-1\right)} + \frac{2\sigma^2}{r\left((1+r)^2-1\right)^2},$$

given that $\dfrac{1}{r(1+r)^2} + \dfrac{2}{r(1+r)^4} + \dfrac{3}{r(1+r)^6} + \cdots$ is an arithmetic-geometric series with

first term equal to $\dfrac{1}{r(1+r)^2}$, difference equal to $\dfrac{1}{r(1+r)^2}$, and ratio equal to

$\dfrac{1}{(1+r)^2}$.

Then, $Var(P_t) = \dfrac{(1+r)^2\sigma^2}{\left((1+r)^2-1\right)^2} + \dfrac{2\sigma^2}{r\left((1+r)^2-1\right)} + \dfrac{2\sigma^2}{r\left((1+r)^2-1\right)^2}.$

Appendix A5

$$E(P_t) = E\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-i} E(D_{t+i}) = \sum_{i=1}^{\infty}(1+r)^{-i}(1+g)^i D_t = \frac{\bar{D}}{r-g}$$

$$Var(P_t) = Var\left(\sum_{i=1}^{\infty}(1+r)^{-i} D_{t+i}\right) = \sum_{i=1}^{\infty}(1+r)^{-2i} Var(D_{t+i}) + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j} Cov(D_{t+i}, D_{t+j}).$$

$Var(P_t)$ is expressed as the sum of two components: the first, $\sum_{i=1}^{\infty}(1+r)^{-2i} Var(D_{t+i})$, is the part of $Var(P_t)$ due to the variability of futu re payoffs, while the second, $2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j} Cov(D_{t+i}, D_{t+j})$, is the part of $Var(P_t)$ due to the co-variabilit y of future different payoffs. The first component is:

$$\sum_{i=1}^{\infty}(1+r)^{-2i} Var(D_{t+i}) = \sum_{i=1}^{\infty}(1+r)^{-2i} \sum_{j=1}^{i}(1+g)^{2(j-1)} \sigma^2 =$$

$$= \sigma^2\left[(1+r)^{-2} + \left((1+r)^{-4}\left(1+(1+g)^2\right)\right) + \left((1+r)^{-6}\left(1+(1+g)^2+(1+g)^4\right)\right) + \cdots\right] =$$

$$= \sigma^2\left[\sum_{i=1}^{\infty}(1+r)^{-2i} + (1+g)^2\sum_{i=2}^{\infty}(1+r)^{-2i} + (1+g)^4\sum_{i=3}^{\infty}(1+r)^{-2i} + \cdots\right] =$$

$$= \sigma^2\left[\frac{(1+r)^{-2}}{1-(1+r)^{-2}} + (1+g)^2\frac{(1+r)^{-4}}{1-(1+r)^{-2}} + (1+g)^4\frac{(1+r)^{-6}}{1-(1+r)^{-2}} + \cdots\right] = \sigma^2\frac{\dfrac{(1+r)^{-2}}{1-(1+r)^{-2}}}{1-(1+g)^2(1+r)^{-2}} =$$

$$= \frac{\sigma^2}{\left((1+r)^2-1\right)\left(1-(1+g)^2(1+r)^{-2}\right)} = \frac{(1+r)^2\sigma^2}{\left((1+r)^2-1\right)\left((1+r)^2-(1+g)^2\right)}.$$

And the second component is:

$$2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j} Cov(D_{t+i}, D_{t+j}) =$$

$$= 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(1+r)^{-i}(1+r)^{-j} E\left[\left(\sum_{k=1}^{i}(1+g)^{i-k}\varepsilon_{t+k}\right)\left(\sum_{l=1}^{j}(1+g)^{j-l}\varepsilon_{t+l}\right)\right] =$$

$$= 2\sigma^2\left[\begin{array}{l}(1+r)^{-1}\left((1+r)^{-2}(1+g)+(1+r)^{-3}(1+g)^2+(1+r)^{-4}(1+g)^3+...\right)+ \\ (1+r)^{-2}\left((1+r)^{-3}\left((1+g)^3+(1+g)\right)+(1+r)^{-4}\left((1+g)^4+(1+g)^2\right)+\cdots\right)+ \\ (1+r)^{-3}\left((1+r)^{-4}\left((1+g)^5+(1+g)^3+(1+g)\right)+(1+r)^{-5}\left((1+g)^6+(1+g)^4+(1+g)^2\right)+\cdots\right)+ \\ +\cdots\end{array}\right] =$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} (1+r)^{-i} (1+r)^{-j} \sum_{k=1}^{i} (1+g)^{j-i+2(k-1)} =$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} (1+r)^{-i} (1+r)^{-j} \frac{(1+g)^{j-i} - (1+g)^{j+i}}{1-(1+g)^2} =$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} (1+r)^{-i} \left((1+r)^{-i-1} \frac{(1+g) - (1+g)^{2i+1}}{1-(1+g)^2} + (1+r)^{-i-2} \frac{(1+g)^2 - (1+g)^{2i+2}}{1-(1+g)^2} + \cdots \right) =$$

$$= 2\sigma^2 \sum_{i=1}^{\infty} (1+r)^{-i} \left(\frac{(1+r)^{-i-1} \frac{(1+g) - (1+g)^{2i+1}}{1-(1+g)^2}}{1-(1+r)^{-1}(1+g)} \right) = 2\sigma^2 \sum_{i=1}^{\infty} \frac{(1+r)^{-2i-1} \left((1+g) - (1+g)^{2i+1} \right)}{\left(1-(1+g)^2 \right) \left(1-(1+r)^{-1}(1+g) \right)} =$$

$$= 2\sigma^2 \left(\sum_{i=1}^{\infty} \frac{(1+r)^{-2i-1}(1+g)}{\left(1-(1+g)^2 \right) \left(1-(1+r)^{-1}(1+g) \right)} - \sum_{i=1}^{\infty} \frac{(1+r)^{-2i-1}(1+g)^{2i+1}}{\left(1-(1+g)^2 \right) \left(1-(1+r)^{-1}(1+g) \right)} \right) =$$

$$= 2\sigma^2 \left(\frac{\frac{(1+r)^{-3}(1+g)}{\left(1-(1+g)^2 \right) \left(1-(1+r)^{-1}(1+g) \right)}}{1-(1+r)^{-2}} - \frac{\frac{(1+r)^{-3}(1+g)^3}{\left(1-(1+g)^2 \right) \left(1-(1+r)^{-1}(1+g) \right)}}{1-(1+r)^{-2}(1+g)^2} \right) =$$

$$= 2\sigma^2 \left(\frac{(1+r)^{-3}(1+g)}{\left(1-(1+g)^2 \right) \left(1-(1+r)^{-1}(1+g) \right) \left(1-(1+r)^{-2} \right)} - \frac{(1+r)^{-3}(1+g)^3}{\left(1-(1+g)^2 \right) \left(1-(1+r)^{-1}(1+g) \right) \left(1-(1+r)^{-2}(1+g)^2 \right)} \right) =$$

$$= 2\sigma^2 \left(\frac{(1+g)}{\left((1+g)^2 -1 \right)(g-r) \left((1+r)^2 -1 \right)} - \frac{(1+g)^3}{\left((1+g)^2 -1 \right)(g-r) \left((1+r)^2 -(1+g)^2 \right)} \right).$$

Then,

$$Var\!\left(P_t | \Omega_t \right) = \frac{(1+r)^2 \sigma^2}{\left((1+r)^2 -1 \right) \left((1+r)^2 -(1+g)^2 \right)} +$$

$$+ \left(\frac{(1+g)^3}{\left((1+g)^2 -1 \right)(r-g) \left((1+r)^2 -(1+g)^2 \right)} - \frac{(1+g)}{\left((1+g)^2 -1 \right)(r-g) \left((1+r)^2 -1 \right)} \right) 2\sigma^2.$$

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